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ω-INTERPOLATIVE ĆIRIĆ-REICH-RUS TYPE CONTRACTIONS IN M-METRIC SPACE

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Abstract: In this paper, we generalize the concept of ω-interpolative Ćirić-Reich-Rus Type Contractions in the framework of M-metric spaces, to find the fixed points and proved some fixed points results for such mappings. Moreover an illustration is provided to support our applicability of obtained results.

Keywords and Phrases: ω-interpolative Ćirić-Reich-Rus Type contraction, M-metric space, ω-orbital.

2020 Mathematics Subject Classification: 47H10, 54H25, 37C25.

1. Introduction and Preliminaries

In 2014, Mehdi Asadi et al. [4] introduced the concept of M-metric space which has a nonzero self distance, as a generalization of metric space. Erdal Karapinar [4] established interpolative contraction to prove existence of fixed points in Metric space. He states that: "For a metric space (X, d), the self mapping $T : X \to X$ is said to be an interpolative kannan type contraction, if there are constants $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \lambda[d(x, Tx)]^{\alpha}[d(y, Ty)]^{1-\alpha},$$

for all $x, y \in X$ with $x \neq Tx$."
In this paper we wield such interpolative contraction using ω- admissible functions. The notion of ω-orbital admissible maps was introduced by Popescu as a refinement

of the concept of α-admissible maps of Samet et al. [15]. Hassen Aydi initiate ω-interpolative Ćirić-Reich-Rus Type in [1] to prove the existence of fixed points in the framework of complete metric space.
In this paper, we generalize the existence of fixed points in M-metric space using ω-interpolative Ćirić-Reich-Rus Type. For more studies of fixed point results for contraction refer [6-14] and reference therein.
In this section we will recall the basic notions of M-metric space and Ćirić-Reich-Rus-Type contraction. The following notion will be used in the presentation,

Definition 1.1. [4] *Let X be a nonempty set. If the function $m : X \times X \to \mathbb{R}^+$ satisfies the following conditions for all $x, y, z \in X$.*

1. $m(x,x) = m(y,y) = m(x,y) \iff x = y,$
2. $m_{xy} \leq m(x,y),$
3. $m(x,y) = m(y,x),$
4. $(m(x,y) - m_{xy}) \leq (m(x,z) - m_{xz}) + (m(z,y) - m_{zy}),$

Then the pair (X, m) is called M-metric space, where

1. $m_{xy} := min\{m(x,x), m(y,y)\}$
2. $M_{xy} := max\{m(x,x), m(y,y)\}.$

Remark 1.2. [4] *For every $x, y \in X$*

1. $0 \leq M_{xy} + m_{xy} = m(x,x) + m(y,y),$
2. $0 \leq M_{xy} - m_{xy} = |m(x,x) - m(y,y)|,$
3. $M_{xy} - m_{xy} \leq (M_{xz} - m_{xz}) + (M_{zy} - m_{zy}).$

Example 1.3. Let $X = [0, \infty)$. Then $m(x,y) = \dfrac{x+y}{2}$ on X is an M-metric space.

Example 1.4. [4] Let (X, m) be an M-metric space. Put

1. $m^w(x,y) = m(x,y) - 2m_{xy} + M_{xy},$
2. $m^s(x,y) = m(x,y) - m_{xy}$ when $x \neq y$ and $m^s(x,y) = 0$ if $x = y$.

Example 1.5. Let $X = \{1, 2, 3\}$ Define $m(1, 1) = 1, m(2, 2) = 9, m(3, 3) = 5, m(1, 2) = m(2, 1) = 10, m(1, 3) = m(3, 1) = 7, m(3, 2) = m(2, 3) = 7$. So (X, m) is M-metric space.

In [4], describes each m metric m on X generates a T_0 topology τ_m on X. The set $\{B_m(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_m(x, \epsilon) = \{y \in X : m(x, y) < m_{xy} + \epsilon\}$, for all $x \in X$ and $\epsilon > 0$, forms a base of τ_m.

Definition 1.6. *Let (X, m) be a M-metric space. Then:*

1. *A sequence $\{x_n\}$ in a M-metric space (X, m) converges to a point $x \in X$ if and only if*
$$\lim_{n\to\infty}(m(x_n, x) - m_{x_n x}) = 0. \tag{1}$$

2. *A sequence $\{x_n\}$ in a $M-$metric space (X, m) is called an $m-$cauchy sequence if*
$$\lim_{n\to\infty}(m(x_n, x_m) - m_{x_n x_m}), \quad \lim_{n\to\infty}(M_{x_n x_m} - m_{x_n x_m})$$
exist.(and are finite)

3. *An M- metric space (X, m) is said to be complete if every m-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_m, to a point $x \in X$ such that*
$$(\lim_{n\to\infty}(m(x_n, x) - m_{x_n x}) = 0 \ \ \& \lim_{n\to\infty}(M_{x_n x} - m_{x_n x}) = 0)$$

Lemma 1.7. [4] *Assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in an M- metric space (X, m). Then*
$$\lim_{n\to\infty}(m(x_n, y_n) - m_{x_n y_n}) = m(x, y) - m_{xy}.$$

Proof. We have
$$|(m(x_n, y_n) - m_{x_n y_n}) - (m(x, y) - m_{xy})| \leq (m(x_n, x) - m_{x_n x}) + (m(y, y_n) - m_{y y_n})$$

Lemma 1.8. [4] *Assume that $x_n \to x$ as $n \to \infty$ in an $M-$ metric space (X, m). Then $\lim_{n\to\infty}(m(x_n, y) - m_{x_n y}) = m(x, y) - m_{xy}$.*

Lemma 1.9. *Assume that $x_n \to x$ and $x_n \to y$ as $n \to \infty$ in an $M-$metric space (X, m). Then $x = y$.*

Lemma 1.10. *Let $\{x_n\}$ be a sequence in an M-metric space (X, m), such that $\exists\ r \in [0, 1)$,*
$$m(x_{n+1}, x_n) \leq r m(x_n, x_{n-1}), \quad \forall n \in \mathbb{N} \tag{2}$$

Then

1. $\lim\limits_{n\to\infty} m(x_n, x_{n-1}) = 0$,

2. $\lim\limits_{n\to\infty} m(x_n, x_n) = 0$,

3. $\lim\limits_{n\to\infty} m_{x_m x_n} = 0$,

4. $\{x_n\}$ *is an* m*-Cauchy sequence.*

Let us denote the set of all non-decreasing self-mapping ψ on $[0, \infty)$ such that:

$$\sum_{n=1}^{\infty} \psi^n(t) < \infty \ for\ each\ t > 0,$$

by Ψ . Further $\psi \in \Psi$, we have $\psi(0) = 0$ and $\psi(t) < t$ for each $t > 0$.

Definition 1.11. *Let* $\omega : X \times X \to [0, \infty)$ *be a mapping and* $X \neq \emptyset$. *A self-mapping* $T : X \to X$ *is said to be an* ω*-orbital admissible if for all* $s \in X$, *we have*

$$\omega(s, Ts) \geq 1 \implies \omega(Ts, Ts^2) \geq 1.$$

Definition 1.12. *Let* (X, d) *be a metric space. The map* $T : X \to X$ *is said to be an* ω*-interpolative Ćirić-Reich-Rus-type contraction if there exist* $\psi \in \Psi$, $\omega : X \times X \to [0, \infty)$ *and positive reals* $\gamma, \beta > 0$, *verifying* $\gamma + \beta < 1$, *such that:*

$$\omega(x, y)d(Tx, Ty) \leq \psi\Big([d(x, y)]^{\beta}.[d(x, Tx)]^{\gamma}.[d(y, Ty)]^{1-\gamma-\beta}\Big) \tag{3}$$

for all $x, y \in X$ $Fix_T(X)$, *where* $Fix_T(X)$ *denotes the set of all fixed points of* T.

2. Main Results

Theorem 2.1. *Suppose a continuous self mapping* $T : X \to X$ *is* ω*-orbital admissible and forms an* ω*-interpolative Ćirić-Reich-Rus-type contraction on a complete* M*-metric space* (X, m). *If there exists* $x_0 \in X$ *such that* $\omega(x_0, Tx_0) \geq 1$, *then* T *has a fixed point in* X.

Proof. Let $x_0 \in X$ be a point such that $\omega(x_0, Tx_0) \geq 1$. Let $\{x_n\}$ be the Picard sequence defined by $x_n = T^n(x_0), n \geq 0$.

If for some n_0, we have $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T, which ends the proof. Otherwise, $x_n \neq x_{n+1}$, for each $n \geq 0$. We have $\omega(x_0, x_1) \geq 1$. Since T is ω-orbital admissible,

$$\omega(x_1, x_2) = \omega(Tx_0, Tx_1) \geq 1.$$

After some iteration

$$\omega(x_n, x_{n+1}) \geq 1 \ for\ all\ n \geq 0.$$

Consider $x = x_n$ and $y = x_{n-1}$
From (2), we have

$$\begin{aligned} m(x_{n+1}, x_n) &\leq \omega(x_n, x_{n-1}) m(Tx_n, Tx_{n-1}) \\ &\leq \psi\Big([m(x_n, x_{n-1})]^\beta.[m(x_n, Tx_n)]^\gamma.[d(x_{n-1}, Tx_{n-1})]^{1-\gamma-\beta}\Big) \\ &= \psi\Big([m(x_n, x_{n-1})]^\beta [m(x_n, x_{n+1})]^\gamma.[m(x_{n-1}, x_n)]^{1-\gamma-\beta}\Big) \\ &= \psi\Big([m(x_n, x_{n+1})]^\gamma.[m(x_{n-1}, x_n)]^{1-\gamma}\Big) \end{aligned}$$

Hence,

$$m(x_{n+1}, x_n) \leq \psi\Big([m(x_n, x_{n+1})]^\gamma.[m(x_{n-1}, x_n)]^{1-\gamma}\Big) \tag{4}$$

By the property of $\psi(t) < t$ for each $t > 0$.

$$\begin{aligned} m(x_{n+1}, x_n) &\leq \psi\Big([m(x_{n-1}, x_n)]^{1-\gamma}.[m(x_n, x_{n+1})]^\gamma\Big) \\ &< [m(x_{n-1}, x_n)]^{1-\gamma}.[m(x_n, x_{n+1})]^\gamma \end{aligned}$$

Therefore

$$m(x_n, x_{n+1}) < m(x_{n-1}, x_n) \ for \ all \ n \geq 1. \tag{5}$$

so that the sequence $\{m(x_{n-1}, x_n)\}$ is decreasing.
Now, we will prove that

$$\lim_{n\to\infty} m(x_{n-1}, x_n) = 0.$$

Suppose that $\lim_{n\to\infty} m(x_{n-1}, x_n) = l, \ where \ l \geq 0.$
From (4), we will get,
$[m(x_{n-1}, x_n)]^{1-\gamma}.[m(x_n, x_{n+1})]^\gamma \leq [m(x_{n-1}, x_n)]^{1-\gamma}.[m(x_{n-1}, x_n)]^\gamma = m(x_{n-1}, x_n)$,
So using (3) and properties of ψ gives
$m(x_{n+1}, x_n) \leq \psi\Big([m(x_n, x_{n+1})]^\gamma.[m(x_{n-1}, x_n)]^{1-\gamma}\Big) \leq \psi(m(x_{n-1}, x_n))$ By repeating this argument,

$$m(x_{n+1}, x_n) \leq \psi(m(x_{n-1}, x_n)) \leq \psi^2(m(x_{n-1}, x_n)) \leq \cdots \leq \psi^n(m(x_{n-1}, x_n)) \tag{6}$$

When $n \to \infty$ in (5) and using the property of Ψ function, $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$, we deduce that $l = 0$, that is

$$\lim_{n\to\infty} m(x_n, x_{n+1}) = 0 \tag{7}$$

Next we will prove that $\{x_n\}$ in m-Cauchy sequence in (X, m).
We have $\lim_{n\to\infty} m(x_n, x_{n+1}) = 0$.
$0 \leq m_{x_n x_{n+1}} \leq m(x_n, x_{n+1}) \implies \lim_{n\to\infty} m(x_n, x_n) = 0.$
and

$m_{x_nx_{n+1}} = \min\{m(x_n, x_n), m(x_{n+1}, x_{n+1})\} \implies \lim_{n\to\infty} m(x_n, x_n) = 0.$
On the other hand,
$m_{x_nx_m} = \min\{m(x_n, x_n), m(x_m, x_m)\} \implies \lim_{n,m\to\infty} m_{x_nx_m} = 0.$
So,

$$\lim_{n,m\to\infty} (M_{x_nx_m} - m_{x_nx_m}) = 0.$$

We will show that

$$\lim_{n,m\to\infty} (m(x_n, x_m) - m_{x_nx_m}) = 0.$$

Define $M^*(x, y) = m(x, y) - m_{xy},\ for\ all\ x, y \in X.$
If $\lim_{n,m\to\infty} M^*(x_n, x_m) \neq 0$, there exist $\epsilon > 0$ and $\{l_k\} \subset \mathbb{N}$ such that

$$M^*(x_{l_k}, x_{n_k}) \geq \epsilon.$$

Suppose that k is the smallest integer which satisfies above equation such that

$$M^*(x_{l_k-1}, x_{n_k}) < \epsilon.$$

By Definition of M-metric space,
$\epsilon \leq M^*(x_{l_k}, x_{n_k}) \leq M^*(x_{l_k}, x_{l_k-1}) + M^*(x_{l_k-1}, x_{n_k}) < M^*(x_{l_k}, x_{l_k-1}) + \epsilon$
Thus

$$\lim_{k\to\infty} M^*(x_{l_k}, x_{n_k}) = \epsilon,$$

which means

$$\lim_{k\to\infty} (m(x_{l_k}, x_{n_k}) - m_{x_{l_k}x_{n_k}}) = \epsilon.$$

On the other hand

$$\lim_{k\to\infty} m_{x_{l_k}x_{n_k}} = 0,$$

so we have

$$\lim_{k\to\infty} m(x_{l_k}, x_{n_k}) = \epsilon. \tag{8}$$

By definition

$$M^*(x_{l_k}, x_{n_k}) \leq M^*(x_{l_k}, x_{l_k+1}) + M^*(x_{l_k+1}, x_{n_k}) + M^*(x_{n_k+1}, x_{n_k}),$$

and

$$M^*(x_{l_k+1}, x_{n_k+1}) \leq M^*(x_{l_k}, x_{l_k+1}) + M^*(x_{l_k}, x_{n_k}) + M^*(x_{n_k+1}, x_{n_k})$$

From the above equations we can conclude that

$$M^*(x_{n_k+1}, x_{n_k}) \leq M^*(x_{l_k}, x_{l_k+1}) \tag{9}$$

taking the limit as $k \to \infty$, together with (6) and (7) we have

$$\lim_{k\to\infty} m(x_{l_k+1}, x_{n_k+1}) = \epsilon.$$

Then, there exists $n_1 \in \mathbb{N}$ such that for all $k \geq n_1$ we have

$$m(x_{l_k}, x_{n_k}) > \frac{\epsilon}{2} \textit{ and } m(x_{l_k+1}, x_{n_k+1}) > \frac{\epsilon}{2} > 0.$$

Since T is ω-orbital, we have

$$\omega(x_{l_k}, x_{n_k}) \geq 1.$$

From the above argument,

$$\omega(x_{l_k}, x_{n_k}) m(Tx_{l_k}, Tx_{n_k}) \leq \psi\Big([m(x_{l_k}, x_{n_k})]^{\beta}[m(x_{l_k}, Tx_{l_k})]^{\gamma}[m(x_{n_k}, Tx_{n_k})]^{1-\gamma-\beta}\Big)$$

when $k \to \infty$,

$$\begin{aligned} \omega(x_{l_k}, x_{n_k}) m(Tx_{l_k}, Tx_{n_k}) &\leq \psi(\epsilon) \\ &< \epsilon. \end{aligned}$$

Since $\omega(x_{l_k}, x_{n_k}) \geq 1$ implies that $m(Tx_{l_k}, Tx_{n_k}) < \epsilon$ which is a contradiction, and therefore $\{x_n\}$ in m-Cauchy sequence. Regarding the completeness of the M-metric space (X, m), we deduce that there is some $x \in X$ so that

$$\lim_{n\to\infty} m(x_n, x) = 0.$$

Since T is continuous, we have $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = T\Big(\lim_{n\to\infty} x_n\Big) = Tx$. Hence the theorem.

In the next theorem, we replace the continuity of T by using the following weakened condition: If $\{a_n\}$ is a sequence in X such that $\omega(a_n, a_{n+1}) \geq 1$ for each $n \in \mathbb{N}$ and $a_n \to a \in X$ as $n \to \infty$, then there exists $\{a_{n(k)}\}$ from $\{a_n\}$ such that $\omega(a_{n(k)}, a) \geq 1$.

Theorem 2.2. *Suppose a self mapping $T : X \to X$ is ω-orbital admissible and forms an ω-interpolative Ćirić-Reich-Rus-type contraction on a complete M-metric space (X, m). Suppose that the above mentioned weakened condition is also fulfilled. If there exists $x_0 \in X$ such that $\omega(x_0, Tx_0) \geq 1$, then T has a fixed point in X.*

Proof. Let $x_0 \in X$ be a point such that $\omega(x_0, Tx_0) \geq 1$. Let $\{x_n\}$ be the Picard

sequence defined by $x_n = T^n(x_0), n \geq 0$.
If for some n_0, we have $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T, which ends the proof. Otherwise, $x_n \neq x_{n+1}$, for each $n \geq 0$. We have $\omega(x_0, x_1) \geq 1$. Since T is ω-orbital admissible,

$$\omega(x_1, x_2) = \omega(Tx_0, Tx_1) \geq 1.$$

After some iteration

$$\omega(x_n, x_{n+1}) \geq 1 \ for \ all \ n \geq 0.$$

Consider $x = x_n$ and $y = x_{n-1}$
From (2), we have

$$\begin{aligned} m(x_{n+1}, x_n) &\leq \omega(x_n, x_{n-1}) m(Tx_n, Tx_{n-1}) \\ &\leq \psi\Big([m(x_n, x_{n-1})]^{\beta}.[m(x_n, Tx_n)]^{\gamma}.[d(x_{n-1}, Tx_{n-1})]^{1-\gamma-\beta}\Big) \\ &= \psi\Big([m(x_n, x_{n-1})]^{\beta}[m(x_n, x_{n+1})]^{\gamma}.[m(x_{n-1}, x_n)]^{1-\gamma-\beta}\Big) \\ &= \psi\Big([m(x_n, x_{n+1})]^{\gamma}.[m(x_{n-1}, x_n)]^{1-\gamma}\Big) \end{aligned}$$

Hence,

$$m(x_{n+1}, x_n) \leq \psi\Big([m(x_n, x_{n+1})]^{\gamma}.[m(x_{n-1}, x_n)]^{1-\gamma}\Big) \tag{10}$$

By the property of $\psi(t) < t$ for each $t > 0$.

$$\begin{aligned} m(x_{n+1}, x_n) &\leq \psi\Big([m(x_{n-1}, x_n)]^{1-\gamma}.[m(x_n, x_{n+1})]^{\gamma}\Big) \\ &< [m(x_{n-1}, x_n)]^{1-\gamma}.[m(x_n, x_{n+1})]^{\gamma} \end{aligned}$$

Therefore

$$m(x_n, x_{n+1}) < m(x_{n-1}, x_n) \ for \ all \ n \geq 1. \tag{11}$$

so that the sequence $\{m(x_{n-1}, x_n)\}$ is decreasing.
Now, we will prove that

$$\lim_{n\to\infty} m(x_{n-1}, x_n) = 0.$$

Suppose that $\lim_{n\to\infty} m(x_{n-1}, x_n) = l, \ where \ l \geq 0.$
From (4), we will get,
$[m(x_{n-1}, x_n)]^{1-\gamma}.[m(x_n, x_{n+1})]^{\gamma} \leq [m(x_{n-1}, x_n)]^{1-\gamma}.[m(x_{n-1}, x_n)]^{\gamma} = m(x_{n-1}, x_n)$,
So using (3) and properties of ψ gives
$m(x_{n+1}, x_n) \leq \psi\Big([m(x_n, x_{n+1})]^{\gamma}.[m(x_{n-1}, x_n)]^{1-\gamma}\Big) \leq \psi(m(x_{n-1}, x_n))$ By repeating this argument,

$$m(x_{n+1}, x_n) \leq \psi(m(x_{n-1}, x_n)) \leq \psi^2(m(x_{n-1}, x_n)) \leq \cdots \leq \psi^n(m(x_{n-1}, x_n)) \tag{12}$$

When $n \to \infty$ in (5) and using the property of Ψ function, $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$, we deduce that $l = 0$, that is

$$\lim_{n\to\infty} m(x_n, x_{n+1}) = 0 \tag{13}$$

By the similar argument of Theorem 2.1, we will prove that the sequence $\{x_n\}$ is Cauchy. Suppose the weakened condition holds. Assume that $x \neq Tx$.
If $x_{n(k)} \neq Tx_{n(k)}$ for each $k > 0$. Due to the condition, there exists a partial subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\omega(x_{n(k)}, x) \geq 1$ for all k. Since $\{m(x_{n(k)}, x)\} \to 0$, $\{m(x_{n(k)}, Tx_{n(k)}\} \to 0$ and $m(x, Tx) > 0$, there is $N \in \mathbb{N}$ such that, for each $k \geq N$,

$$m(x_{n(k)}, x) \leq m(x, Tx) \textit{ and } m(x_{n(k)}, Tx_{n(k)}) \leq m(x, Tx).$$

Take $x = x_{n(k)}$ and $y = x$ in (2), we get that:

$$\begin{aligned} m(x_{n(k)+1}, Tx) &\leq \omega(x_{n(k)}, x) m(Tx_{n(k)}, Tx) \\ &\leq \psi([m(x_{n(k)}, x)]^{\beta}.[m(x_{n(k)}, Tx_{n(k)})]^{\gamma}[m(x, Tx)]^{1-\gamma-\beta}) \\ &\leq \psi([m(x, Tx)]^{\beta}.[m(x, Tx)]^{\gamma}[m(x, Tx)]^{1-\gamma-\beta}) \\ &\leq \psi(m(x, Tx)). \end{aligned}$$

Letting $k \to \infty$, we find that :

$$0 < m(x, Tx) \leq \psi(m(x, Tx)) < m(x, Tx),$$

which is a contradiction. Thus, $x = Tx$. Hence the theorem.

Corollary 2.3. *Let T be a self mapping on a complete M-metric space (X, m) such that:*

$$\omega(x, y) d(Tx, Ty) \leq \psi\Big([d(x, y)]^{\beta}.[d(x, Tx)]^{\gamma}.[d(y, Ty)]^{1-\gamma-\beta}\Big)$$

for all $x, y \notin Fix_T(X)$, where $\gamma.\beta$ are positive reals verifying $\gamma + \beta < 1$ and $\omega(x, y) = 1$. Then, T has a fixed point in X.

In the next corollary we consider $\psi(t) = \lambda t \;\; for\ some\ \lambda \in [0, 1)$.

Corollary 2.4. *Let T be a self mapping on a complete M-metric space (X, m) such that:*

$$\omega(x, y) d(Tx, Ty) \leq \lambda\Big([d(x, y)]^{\beta}.[d(x, Tx)]^{\gamma}.[d(y, Ty)]^{1-\gamma-\beta}\Big)$$

for all $x, y \notin Fix_T(X)$, where $\gamma.\beta$ are positive reals verifying $\gamma + \beta < 1$ and $\lambda \in [0, 1)$., and $\omega(x, y) = 1$. Then, T has a fixed point in X.

3. Consequences

In this section, we endeavor to learn existence and uniqueness of fixed points in $M-$metric space through example which supports our result.

Example 3.1. Consider $X = [0, 1]$ and $m : X \times X \to [0, \infty)$ with $m(x, y) = \frac{x+y}{2}$.

Then the mapping $T : X \to X$ is defined as $T(x) = \begin{cases} \frac{x^2}{3} & if\ x \in \left[0, \frac{1}{2}\right] \\ 1 & if\ x \in \left(\frac{1}{2}, 1\right] \end{cases}$ and

$\omega(x, y) = \begin{cases} 2\ if\ x, y \in \left[0, \frac{3}{4}\right] \\ 1 \quad otherwise. \end{cases}$ with $\psi(t) = t$ where $\psi \in \Psi$.

In order to support our result, we will discuss this in three cases:

Case I: If $x, y \in \left[0, \frac{1}{2}\right]$ then

$$\begin{aligned}
\omega(x,y)m(Tx,Ty) &\leq \psi\Big([m(x,y)]^{\beta}.[m(x,Tx)]^{\gamma}.[m(y,Ty)]^{1-\gamma-\beta}\Big) \\
2m(\frac{x^2}{3}, \frac{y^2}{3}) &\leq \psi\Big([\frac{x+y}{2}]^{\beta}.[m(x, \frac{x^2}{3})]^{\gamma}.[m(y, \frac{y^2}{3})]^{1-\gamma-\beta}\Big) \\
&\leq \psi\Big(\Big[\frac{x+y}{2}\Big]^{\beta}.\Big[\frac{3x+x^2}{6}\Big]^{\gamma}.\Big[\frac{3y+y^2}{6}\Big]^{1-\gamma-\beta}\Big) \\
2\frac{x^2+y^2}{6} &\leq \psi\Big(\Big[\frac{3x+3y}{6}\Big]^{\beta}.\Big[\frac{3x+x^2}{6}\Big]^{\gamma}.\Big[\frac{3y+y^2}{6}\Big]^{1-\gamma-\beta}\Big) \\
\frac{x^2+y^2}{3} &\leq \frac{1}{6}3^{\beta}[x+y]^{\beta}.[3x+x^2]^{\gamma}.[3y+y^2]^{1-\gamma-\beta}
\end{aligned}$$

Above inequality is always true whenever $\gamma + \beta < 1$ for all $x, y \in \left[0, \frac{1}{2}\right]$.

Case II: If $x \in \left[0, \frac{1}{2}\right]$ *and* $y \in \left(\frac{1}{2}, 1\right]$

$$\begin{aligned}
\omega(x,y)m(Tx,Ty) &\leq \psi\Big([m(x,y)]^{\beta}.[m(x,Tx)]^{\gamma}.[m(y,Ty)]^{1-\gamma-\beta}\Big) \\
1.m(\frac{x^2}{3}, 1) &\leq \psi\Big([m(x,y)]^{\beta}.[m(x, \frac{x^2}{3})]^{\gamma}.[m(y,1)]^{1-\gamma-\beta}\Big) \\
\frac{x^2+3}{6} &\leq \psi\Big(\Big[\frac{x+y}{2}\Big]^{\beta}.\Big[\frac{3x+x^2}{6}\Big]^{\gamma}.\Big[\frac{y+1}{2}\Big]^{1-\gamma-\beta}\Big)
\end{aligned}$$

When $\gamma + \beta < 1$, then T has a fixed point.
Case III:
If $x, y \in \left(\frac{1}{2}, 1\right]$ then $\omega(x, y) = 1$ then (2) reduces to Interpolative Ćirić-Reich-Rus-Type contractions which guarantees fixed points for T.
So, $Fix_T(X) = \{0, 1\}$.

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